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Extended Persistence

Bachelor's thesis

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1 Introduction

The method of persistent homology was introduced around the change of the millennium to measure topological artifacts of a given shape. This measurement could then be used to smoothen the space by removing disturbances with small persistence and to compare shapes and spaces with more detail than just their topology. Applications arose for example in biology, specifically protein docking, and computer sciences, specifically image recognition.

Only recently in 2009, the concept of persistent homology was extended to include essential homology classes by [CSEH09] and thus enabled the measurement of essential topological features in contrast to auxiliary artifacts as well. However, this extension was motivated by application and was done explicitly only for manifolds, while the original persistence theory works on other types of spaces as well.

In this thesis we will give a brief introduction to the requirements of the topic, including homology theory, manifolds and Morse theory, and then develop the concept of persistence and extended persistence on manifolds, giving proofs where they are missing in other literature and illustrating the process with examples. We then generalize the definition of extended persistence to simplicial complexes to make this method applicable in more situations and study which properties survive this generalization and which do not.

2 Prerequisites

In this section, we will introduce some fundamental concepts that are needed later on. Mainly, we will give a brief introduction to homology theory, define manifolds and state Poincaré and Lefschetz duality.

2.1 Simplicial homology

Homology is a method of describing some topological features like holes and torsions of a space. It can be quite efficiently computed for given spaces, in contrast to other topological groups like fundamental groups.

There are several different homology theories, all following the same axioms. We will use only one of those theories, simplicial homology, which applies to objects consisting of triangle-like parts. The definitions are adapted from [Mun84, ch. 1] if not stated otherwise.

Definition (Simplicial complex)**.** First of all, we define an *n-simplex* (or *simplex of dimension n*) with $n \in \mathbb{N}$ to be the convex hull of $n+1$ points $\{a_0, a_1, \ldots, a_n\} \subset \mathbb{R}^N$ $(N \geq n)$ in general position, consisting of all points

$$x = \sum_{i=0}^{n} t_i a_i \quad \text{with } t_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=0}^{n} t_i = 1.$$

We call this simplex *spanned by* the *vertices* $a_0, a_1, \ldots, a_n$, denote it by $(a_0, a_1, \ldots, a_n)$ and call the simplices spanned by subsets of these points (with $k+1$ elements) the *(k-)faces* of the simplex.

Finally, a *simplicial complex* is a finite set K of simplices, satisfying the following two conditions:

(SC1) All faces of a simplex in K are also in K.

(SC2) The intersection of two simplices in K is a face of both (and thus contained in K itself).

Any subset of K which itself satisfies the requirements is called a *subcomplex* of K. To emphasize the underlying space rather than the complex structure, we sometimes write $|K|$.

Definition (Homology group)**.** Let K be a simplicial complex and let K_p be the set of p-simplices in K.

We need to define an *orientation* on its simplices, although we can choose it arbitrarily: We consider two orderings of the vertices of a simplex equivalent if they differ by an even permutation. The two equivalence classes that arise from this relation are called the *orientations* of the simplex (except for the zero-dimensional simplex, consisting of only one point, which has only one orientation). We denote the simplex $(a_0, a_1, \ldots, a_n)$ together with its orientation by $[a_0, \ldots, a_n]$ and its opposite orientation by $-[a_0, a_1, \ldots, a_n]$. By abuse of notation, we will call the set of all (p-dimensional) oriented simplices K (respectively K_p) as well.

A *p-chain* is a map c of the oriented p-simplices of K to a unitary ring $\mathbb{K}$ with the following properties:

(CH1) $c(-\sigma) = -c(\sigma)$ for every oriented simplex σ in K_p.

(CH2) $\operatorname{supp}(c) = \{\sigma \in K_p \mid c(\sigma) \neq 0\}$ is finite.

The set of all p-chains over K is called $C_p(K; \mathbb{K})$. It is the free abelian group with *elementary chains* as a basis. Those are p-chains that map one oriented simplex $[a_0, a_1, \ldots, a_p]$

to 1, its opposite orientation to -1 and everything else to 0, denoted by $[a_0 : a_1 : \ldots : a_p]$. The group operation is of course the addition of function values. This enables us to write every p-chain c as the formal sum

$$c = \sum_{[a_0:a_1:\ldots:a_p]\in K_p} c([a_0 : a_1 : \ldots : a_p]) \cdot [a_0 : a_1 : \ldots : a_p]$$

We then define the *boundary operator* of dimension p ($p \geq 1$) as the map

$$\begin{aligned} \partial_p : C_p(K;\mathbb{K}) &\to C_{p-1}(K;\mathbb{K}) \\ [a_0 : a_1 : \ldots : a_p] &\mapsto \sum_{i=0}^{p} [a_0 : a_1 : \ldots : \hat{a}_i : \ldots : a_p]. \end{aligned}$$

expanded to all p-chains, where $\hat{a}_i$ denotes the omission of a_i.

Finally, we call the kernel of ∂_p the group of *p-cycles*, denoted $Z_p(K;\mathbb{K})$, and the image of ∂_{p+1} the group of *p-boundaries*, denoted $B_p(K;\mathbb{K})$. Since $\partial_p \circ \partial_{p+1} = 0$, we define the *p-th homology group* as

$$H_p(K;\mathbb{K}) := Z_p(K;\mathbb{K})/B_p(K;\mathbb{K})$$

This group is a free abelian group itself and its rank is called the *p-th Betti number* (of K).

Two p-chains $\alpha, \beta \in C_p(K;\mathbb{K})$ are called *homologous* if they only differ by a p-boundary, thus if $\alpha = \beta + \gamma$ with $\gamma \in B_p(K;\mathbb{K})$.

If the choice of $\mathbb{K}$ is obvious, we can omit it in the notation and write $C_p(K)$, $Z_p(K)$, $B_p(K)$ and $H_p(K)$.

Most often in other applications, $\mathbb{K}$ is the ring of integers $\mathbb{Z}$, but any ring with 1 suffices. For the topic of persistent homology, we will use $\mathbb{Z}/2\mathbb{Z} =: \mathbb{Z}_2$ exclusively. The reason for that will be pointed out later on. This also has the advantage that the homology groups are not only groups but vector spaces (over $\mathbb{Z}_2$) and we can talk about their dimension instead of their rank. Therefore, we will almost always omit the explicit mention of $\mathbb{K}$.

It is worth a note that an inclusion of one simplicial complex into another one induces a homomorphism on the homology groups:

Lemma 2.1 (Induced homomorphism)**.** *Let K, L be two simplicial complexes with $K \subseteq L$. Then the inclusion map $i : K \hookrightarrow L$ induces a homomorphism $H_p(K) \to H_p(L)$ for each dimension p.*

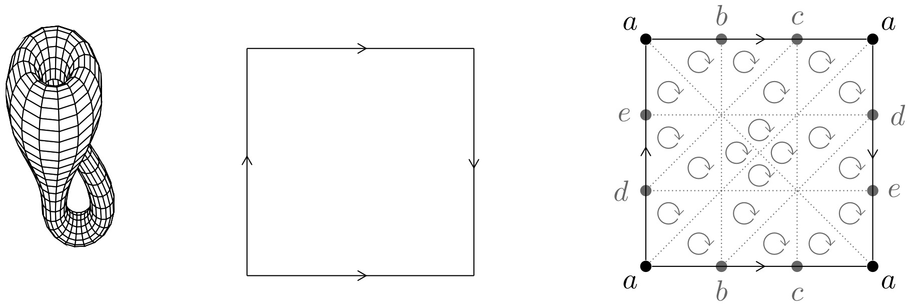

Figure 1: The Klein bottle along with its representation as a rectangle with its edges identified. Also pictured is a possible triangulation with orientations of their 2-simplices, adapted from [Mun84, fig. 6.6].

Proof. The inclusion i induces homomorphisms on the chain groups simply by including the vertices of the elementary chains:

$$\begin{aligned} C_p(K) &\to C_p(L) \\ [a_0 : a_1 : \ldots : a_p] &\mapsto [i(a_0) : i(a_1) : \ldots : i(a_p)]. \end{aligned}$$

Thus the following diagram commutes and gives us a homomorphism $H_p(K) \to H_p(L)$:

$$\begin{array}{ccc} Z_p(K) & \longrightarrow & Z_p(L) \\ \downarrow & & \downarrow \\ Z_p(K)/B_p(K) & \longrightarrow & Z_p(L)/B_p(L) \end{array}$$

□

Example 2.2. We will visualize the features of homology groups by computing them for a well known topological object, the Klein bottle, as done in [Mun84, thm. 6.3]. We will therefore represent it as a rectangle with its edges glued together, one pair straight and one pair twisted, as pictured in figure 1.

We will denote the whole triangulation complex by L and its boundary by B and orient each 2-simplex of L counter-clockwise. The orientation of the 1-simplices can be chosen arbitrarily. For convenience, we will define

$$\begin{aligned} \alpha &:= [a : b] + [b : c] + [c : a], \\ \beta &:= [a : d] + [d : e] + [e : a], \\ \gamma &:= \sum_{\sigma \in C_2(L)} \sigma. \end{aligned}$$

We will compute the homology using the following observations:

(1) Every 1-cycle of L is homologous to a 1-cycle carried by B.

(2) Every 1-cycle of L carried by B is of the form $m \cdot \alpha + n \cdot \beta$ with $m, n \in \mathbb{K}$.

(3) For every 2-chain δ of L whose boundary $\partial\delta$ is carried by B, there is some $q \in \mathbb{K}$ such that $\delta = q \cdot \gamma$.

(4) $\partial\gamma = 2\beta$.

The first observation is immediate: Every 1-simplex a of a 1-cycle of L can be "pushed outwards" by the boundary of a 2-simplex having a as a face and with its third vertex "nearer" to B.

The second observation is almost as immediate: Let σ be a 1-cycle on B. That means that $\partial\sigma = 0$ and thus $(\partial\sigma)(x) = 0$ for each $x \in \{a, b, c, d, e\}$. This equation evaluated for $x = b$ leads to $\sigma[a : b] = \sigma[b : c]$, evaluated for $x = c$ leads to $\sigma[b : c] = \sigma[c : a]$ and analogously $\sigma[a : d] = \sigma[d : e] = \sigma[e : a]$.

At this point we already know that for computing $H_1(L)$, we only need to consider 1-cycles carried by B and that these are of the form $m \cdot \alpha + n \cdot \beta$. Let us look at the 1-boundaries:

For our third observation let δ be any 2-chain of L with boundary $\partial\delta$ carried by B. If any oriented 2-simplex σ had another value under δ than its neighbor, their common face σ would have non-zero value under $\partial\delta$ and thus $\partial\delta$ would not be carried by B.

To verify the fourth and last observation, we resort to plain computation, exploiting that every 1-simplex not carried by B is a face of exactly two (oriented) 2-simplices and is not contained in the boundary of their sum, thus does not need to be counted. So we get:

$$\begin{aligned}\partial\gamma = \sum_{\sigma \in C_2(L)} \partial\sigma &= [a : d] + [d : e] + [e : a] + [a : b] + [b : c] + [c : a] \\ &\quad + [a : d] + [d : e] + [e : a] + [a : c] + [c : b] + [b : a] \\ &= 2([a : d] + [d : e] + [e : a]) = 2\beta.\end{aligned}$$

Now we know everything there is to know about $H_1(L)$: Only 1-cycles carried by B are relevant, they are always of the form $m \cdot \alpha + n \cdot \beta$ and of these, those of the form $2n \cdot \beta$ are killed by $\partial\sigma$.

While we said earlier we will use $\mathbb{K} = \mathbb{Z}_2$ throughout the examples, we will only this one time look at the difference if we use $\mathbb{Z}$ instead: For $\mathbb{K} = \mathbb{Z}$, the fourth observation is absolutely correct, but in the case of $\mathbb{K} = \mathbb{Z}_2$, it is misleading, because $2\beta = 0\beta = 0$. This of course makes a big difference, as we are inspecting kernels.

So we get two different results, depending on the choice of $\mathbb{K}$:

$$H_1(L;\mathbb{Z}) \cong (\mathbb{Z}\oplus\mathbb{Z})/(0\oplus 2\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}_2 = \mathbb{K}\oplus\mathbb{Z}_2 \qquad H_2(L;\mathbb{Z}) \cong 0$$
$$H_1(L;\mathbb{Z}_2) \cong \mathbb{Z}_2\oplus\mathbb{Z}_2 = \mathbb{K}\oplus\mathbb{K} \qquad H_2(L;\mathbb{Z}_2) \cong \mathbb{Z}_2 = \mathbb{K}$$

We have an information loss at two places when using $\mathbb{Z}_2$ instead of $\mathbb{Z}$: First, we cannot distinguish between the two factors of $H_1(L;\mathbb{Z}_2)$ whereas they have different meanings in $H_1(L;\mathbb{Z})$. Second, and more obviously, the second homology group is not trivial.

Contentwise, we lose the information of orientability of the 2-manifold of the Klein bottle (or in this case non-orientability).

2.2 Relative homology and cohomology

Based on homology and its idea, we can construct new groups: What if we are interested in a simplicial complex in general, but do not care about a specific part of it? That's what relative homology deals with:

Definition (Relative homology)**.** Given a simplicial complex K and a subcomplex $K_0 \subseteq K$, we define the *group of relative chains* as

$$C_p(K,K_0) := C_p(K)/C_p(K_0).$$

It's easy to see that the standard boundary operator $\partial_p : C_p(K) \to C_{p-1}(K)$ along with its restricted version $\partial_p|_{C_p(K_0)} : C_p(K_0) \to C_{p-1}(K_0)$ induces a *relative boundary operator*

$$\partial_p : C_p(K,K_0) \to C_{p-1}(K,K_0).$$

Analogously to the standard homology, the *relative homology* is then defined as

$$H_p(K,K_0) := \operatorname{Ker}\partial_p / \operatorname{Im}\partial_{p+1}.$$

A note on notation: Of course, relative homology can use different coefficient rings as well. When omitted we assume $\mathbb{K} = \mathbb{Z}_2$. However, $H_p(K,K_0)$ is not to be confused with $H_p(K;\mathbb{K})$.

Lemma 2.3 (Excision theorem, [Mun84, thm. 9.1])**.** *Let K be a simplicial complex and K_0 a subcomplex. If $U \subseteq |K_0|$ is an open subset such that $|K| \setminus U$ is the underlying of a* *subcomplex of K, denoted L, then for the subcomplex L_0 of K whose polytope is $K_0 \setminus U$ there is an isomorphism*

$$H_p(L, L_0) \cong H_p(K, K_0).$$

After the introduction of homology, another similar concept was proposed that is used nowadays in parallel to homology: Cohomology. While it is very similar in construction, it has the disadvantage of being less intuitive, but instead brings along some algebraic advantages we will not discuss here.

Definition (Cohomology)**.** We recall that for any two abelian groups A and G, the set $\operatorname{Hom}(A, G)$ of homomorphisms from A to G is an abelian group itself, adding homomorphisms by adding their values.

Given a simplicial complex K with its chain groups $C_p(K; \mathbb{K})$, we define the *group of cochains* along with the *coboundary operator* and the *p-th cohomology group* of K:

$$\begin{aligned} C^p(K; \mathbb{K}) &:= \operatorname{Hom}(C_p(K; \mathbb{K}), \mathbb{K}) \\ \delta : C^p(K; \mathbb{K}) &\to C^{p+1}(K; \mathbb{K}) \\ f &\mapsto f \circ \partial_{p+1} \\ H^p(K; \mathbb{K}) &:= \operatorname{Ker} \delta_p / \operatorname{Im} \delta_{p-1} \end{aligned}$$

where ∂_p is the usual boundary operator.

Note that the coboundary operator *increases* dimension by one, in contrast to the boundary operator, which decreases it.

We can apply relative homology to cohomology just as well and denote it by $H^p(K, K_0; \mathbb{K})$. While we do not need excision for our purposes, we do need homomorphisms induced by inclusion (cf. Lemma 2.1).

Lemma 2.4 (Induced homomorphism)**.** *Let K, L be two simplicial complexes with $K \subseteq L$. Then the inclusion map $i : K \hookrightarrow L$ induces a homomorphism $j : H^p(L; \mathbb{K}) \to H^p(K; \mathbb{K})$ for each dimension p.*

Proof. Comparable to the proof for Lemma 2.1, we only need to find the homomorphism for the cochain groups. Any element $f \in C^p(L; \mathbb{K})$ is a map $C_p(L; \mathbb{K}) \to \mathbb{K}$, so we only need to concatenate our induced homomorphism $\varphi : C_p(K; \mathbb{K}) \to C_p(L; \mathbb{K})$ from Lemma 2.1 with f to get a map

$$f \circ \varphi : C_p(K; \mathbb{K}) \to \mathbb{K}.$$

This map is the image of f under our homomorphism in question j, as it is an element of $C^p(K; \mathbb{K})$. □

Note that because we dualize the whole setting, for inclusion we receive homomorphisms from the larger to the smaller set.

2.3 Manifolds and their (Co)homology

We will at first focus on surfaces and their higher-dimensional equivalents: manifolds.

Definition (Manifold)**.** A topological space $\mathbb{M}$ is called an *(n-)manifold (without boundary)* iff for each point $x \in \mathbb{M}$ there is an open neighborhood $U_x \ni x$ and a homeomorphism $U_x \to O_x$ to and open subset $O_x \in \mathbb{R}^n$.

Likewise, $\mathbb{M}$ is called an *(n-)manifold with boundary* if such neighborhoods and homeomorphisms exist for open subsets of the euclidean half-space $\{x \in \mathbb{R}^n \mid x_n \geq 0\}$.

Definition (Triangulability)**.** A topological space X is triangulable, if there is a simplicial complex K and a homeomorphism $X \to K$.

Not every manifold is triangulable and not every triangulable space is a manifold. Those manifolds that are triangulable inherit the properties we find in simplicial complexes, in particular, we can calculate their simplicial homology and cohomology.

Apart from being comparatively easy to imagine (at least up to dimension 3), manifolds have a big advantage over other topological spaces: A manifold's homology and cohomology are linked by the following duality theorems:

Theorem 2.5 (Lefschetz duality, [Hat12, thm. 3.43])**.** *Let M be a compact, triangulable n-manifold with compact boundary ∂M. Then there are isomorphisms*

$$H_p(M; \mathbb{Z}_2) \cong H^{n-p}(M, \partial M; \mathbb{Z}_2)$$

for every dimension p.

In particular, for manifolds without boundary, this theorem yields another, older duality:

Corollary 2.6 (Poincaré duality)**.** *Let M be a compact n-manifold without boundary. Then there are isomorphisms*

$$H_p(M; \mathbb{Z}_2) \cong H^{n-p}(M; \mathbb{Z}_2)$$

for every dimension p.

3 Persistence on manifolds

While the homology of an object tells us much about its general shape, two homologically equivalent spaces can be quite different in application. Consider for example a coffee cup with a handle and a donut. They are homeomorphic, but rarely considered to have the same shape in application.

Furthermore, homology does not consider the size of the space's features. A torus with a small hole is the same as one with a big hole. While for some areas of application this is a desired property, other areas need to *measure* these features. And that's where persistence comes in: Rather than only inspecting the space as a whole, we consider a filtration of the space, that is, we construct it step by step, and then look at the homology at each step.

An introduction to the topic that is based more on intuition and less on formalism can be found in [EH10, ch. VII].

3.1 Morse theory

While Morse theory, the inspection of manifolds through differentiable functions on them, is a whole area of study all by itself, we will only need a few parts of it. For an overview over the theory see for example [Mil73].

Definition (Critical value)**.** Let $\mathbb{M}$ be an n-manifold and $f : \mathbb{M} \to \mathbb{R}$ a smooth map. We then call $x \in \mathbb{M}$ a *critical point* of f iff its differential at x vanishes ($Df_x = 0$). The value $f(x)$ of a critical point is called a *critical value*.

Such a critical point x of a function f is called *non-degenerate* iff its Hessian matrix

$$H(f)(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{1 \leq i,j \leq n}$$

is non-singular (and vice versa, it is degenerate iff the matrix is singular).

The concept of critical points is the higher-dimensional generalization of local minima, maxima and saddle points of the graph of one-dimensional functions.

Theorem 3.1 ([Mil73, thm. 3.1])**.** *Let $f : \mathbb{M} \to \mathbb{R}$ be a smooth function on a manifold $\mathbb{M}$. For any $a < b$ such that $f^{-1}([a, b])$ is compact and does not contain a critical point of f, the sublevel sets*

$$\mathbb{M}_a := f^{-1}((-\infty, a]) \quad \textit{and} \quad \mathbb{M}_b := f^{-1}((-\infty, b])$$

are diffeomorphic (and in particular homeomorphic).

This tells us that homology of sublevel sets $f^{-1}((-\infty, a])$ of a differentiable function only changes while varying a when a passes a critical value of f. We can even describe, how the homology changes. Therefore, we will cite a theorem from [Mil73], even though some of the terms (index and cell) used were not defined in this paper:

Theorem 3.2 ([Mil73, thm. 3.2])**.** *Let $f : \mathbb{M} \to \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point (with index λ). Setting $f(p) = c$, suppose that $f^{-1}([c - \varepsilon, c + \varepsilon])$ is compact, and contains no critical point of f other than p, for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $\mathbb{M}_{c+\varepsilon} := f^{-1}((-\infty, c + \varepsilon])$ has the homotopy type of $\mathbb{M}_{c-\varepsilon} := f^{-1}((-\infty, c - \varepsilon])$ with a λ-cell attached.*

Corollary 3.3. *Given the situation from the theorem, $H_p(\mathbb{M}_{c+\varepsilon}) \cong H_p(\mathbb{M}_{c-\varepsilon})$ for all but one dimension p. In this last dimension, we have $\operatorname{rank} H_p(\mathbb{M}_{c+\varepsilon}) = \operatorname{rank} H_p(\mathbb{M}_{c-\varepsilon}) \pm 1$.*

Proof of corollary. The attaching of a λ-cell changes homology in only one dimension by at most one rank. Compare [Lee11, prop. 13.33]. □

However, critical values may come in bunches: Consider for example the function $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2$, where the whole y axis consists of critical points (cf. figure 2 on page 12).

Lemma 3.4 (Lemma of Morse, [Mil73, lem. 2.2])**.** *Let p be a non-degenerate critical point of $f : \mathbb{M} \to \mathbb{R}$ for some d-manifold $\mathbb{M}$. Then there is a local coordinate system $(y_1, y_2, \ldots, y_d)$ for a neighborhood U of p such that $y_k(p) = 0$ for all $k = 1, 2, \ldots, d$ and*

$$f(x) = f(p) - y_1(x)^2 - y_2(x)^2 - \ldots - y_\lambda(x)^2 + y_{\lambda+1}(x)^2 + y_{\lambda+2}(x)^2 + \ldots + y_n(x)^2$$

for some $\lambda \in \{0, 1, \ldots, d\}$ and all $x \in U$.

In particular, non-degenerate critical points are isolated, meaning that there is a neighborhood around each non-degenerate critical point without any other critical points.

Proof. For the first part of the lemma, see [Mil73, lem. 2.2].

The second part follows easily: For any point $q \in U$ in the aforementioned neighborhood, the partial derivative along one of the local coordinates y_k is

$$\begin{aligned}\frac{\partial}{\partial y_k(q)} f(q) &= \frac{\partial}{\partial y_k(q)} f(p) - \sum_{i=1}^{\lambda} \frac{\partial}{\partial y_k(q)} \left(y_i(q)^2\right) + \sum_{i=\lambda+1}^{n} \frac{\partial}{\partial y_k(q)} \left(y_i(q)^2\right) \\ &= 0 \pm \frac{\partial}{\partial y_k(q)} \left(y_k(q)^2\right) \\ &= \pm 2 y_k(q)\end{aligned}$$

So for $q \neq p$, at least for one $k \in \{1, \dots, n\}$ we have a non-zero coordinate $y_k(q)$ and thus a non-zero partial derivative. Thus, there can be no other critical points in U. □

Definition (Morse function)**.** Let $f : \mathbb{M} \to \mathbb{R}$ be a smooth function on a manifold $\mathbb{M}$. f is called a *Morse function* if it has no degenerate critical points.

We will consider Morse functions, as they only have isolated critical points as per Lemma 3.4. That makes it easy to talk about the points where the homology changes when increasing the threshold $a \in \mathbb{R}$ for the sublevel sets $\mathbb{M}_a$, as they change only at these isolated points as per Lemma 3.1. Furthermore, if we look at compact manifolds, the second part of Lemma 3.4 gives us that there are only finitely many critical points.

Example 3.5. We will often consider n-manifolds $\mathbb{M}$ embedded in $\mathbb{R}^{n+1}$, and the most intuitive Morse function on such a manifold may be its *height function* relative to a baseplane, represented by the scalar product of $x \in \mathbb{M}$ with the baseplane's normal vector h:

$$\begin{aligned} f : \mathbb{M} &\to \mathbb{R} \\ x &\mapsto \langle x, h \rangle \end{aligned}$$

We will quickly see that not all height functions are Morse functions: Our earlier non-Morse example $(x, y) \mapsto x^2$ can be easily extended to the height function of the graph of this very function (cf. figure 2 on the following page) relative to the x-y-plane by adding a third coordinate: $(x, y, z) \mapsto x^2$. However, if we chose the y-z-plane as the baseplane, we get $(x, y, z) \mapsto y$ for the very same manifold, having no critical points at all and thus being a Morse function.

3.2 Ordinary Persistence

Definition (Persistent homology groups)**.** Let $\mathbb{M}$ be a triangulable, compact d-manifold and let $f : \mathbb{M} \to \mathbb{R}$ be a Morse function with critical values $a_1 < a_2 < \ldots < a_n$.

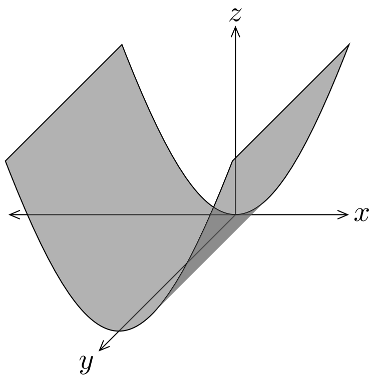

Figure 2: Height function demonstrated on the graph of $(x, y) \mapsto x^2$.

We then choose interleaved values

$$b_0 < a_1 < b_1 < \ldots < a_n < b_n$$

and inspect the sublevel sets $\mathbb{M}_i := f^{-1}((-\infty, b_i])$ and superlevel sets $\mathbb{M}^i := f^{-1}([b_i, \infty))$. The choice of the b_i does not affect the homology of these sets because of theorem 3.1.

These sub- and superlevel sets are obviously nested by inclusion, so we have induced homomorphisms on their homology groups:

$$0 = H_p(\mathbb{M}_0) \to H_p(\mathbb{M}_1) \to \ldots \to H_p(\mathbb{M}_n) = H_p(\mathbb{M}).$$

By skipping steps we get homomorphisms $f_p^{i,j} : H_p(\mathbb{M}_i) \to H_p(\mathbb{M}_j)$ for each pair $0 \leq i \leq j$, where $H_p^{i,i} := H_p(\mathbb{M}_i)$. We will call the images of these homomorphisms $H_p^{i,j} := \mathrm{Im} f_p^{i,j}$ the *p-th persistent homology groups* and their ranks respectively the *p-th persistent Betti numbers* $\beta_p^{i,j}$.

Definition (Birth, Death and Persistence)**.** Given persistent homology groups $H_p^{i,j}$ coming from a Morse function f, we can observe that a homology class $\gamma \in H_p(K_i)$ is *born* at $\mathbb{M}_i$ iff $\gamma \notin H_p^{i-1,i}$. In contrast, if it is born at $\mathbb{M}_i$, it *dies* entering $\mathbb{M}_j$ exactly when it "merges" with an older (earlier born) class, meaning that $f_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $f_p^{i,j}(\gamma) \in H_p^{i-1,j}$.

We set the *persistence* of a class $\gamma \in H_p(\mathbb{M}_i)$ born at $\mathbb{M}_i$ that dies at $\mathbb{M}_j$ to

$$\mathrm{pers}(\gamma) := f(a_j) - f(a_i).$$

Sometimes, we only look at the indices rather than the function values themselves, so we set the *index persistence* of γ to

$$\mathrm{ipers}(\gamma) := j - i.$$

Furthermore, we define

$$\mu_p^{i,j} := (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

to count the cosets of homology classes that are born at $\mathbb{M}_i$ and die entering $\mathbb{M}_j$.

To present these persistences in a visually appealing way, we use *persistence diagrams*: For a fixed dimension p, we draw a point at (a_i, a_j) (or at (i, j) if we are inspecting index persistence) in the real plane $\mathbb{R}^2$ if $\mu_p^{i,j} \neq 0$. As there may be more than one such class, these points are actually multipoints with multiplicity $\mu_p^{i,j}$. The collection of these multipoints and their multiplicities is the *persistence diagram* of the Morse function f, denoted by $\mathrm{Dgm}_p(K_*)$. The (index) persistence of the classes leading to each multipoint can be found in the diagram as the vertical distance to the main diagonal.

An example for all these definitions will be given later on the next page, a persistence diagram can be seen in figure 3 on page 15.

We silently conveyed the impression that a homology class doesn't change between it's birth and death. This is correct because we initially assumed $\mathbb{K} = \mathbb{Z}_2$, which does not leave much space for variation of the class, but for other coefficient rings, this may be different: If for example we were to compute the persistent homology groups of the Klein bottle with coefficients from $\mathbb{Z}$, we could have one step where $\mathbb{Z}$ is mapped to $\mathbb{Z}_2$. This would lead to the birth of the homology class $\mathbb{Z}$ being paired with the death of half the class, $2\mathbb{Z}$. To avoid such situations (which might even lead to one birth being paired with more than one death), we use $\mathbb{Z}_2$ as coefficients. Otherwise we would have to require orientability of the manifold, which would lead to the same definitions but on a smaller range of objects.

We will look at the definition of $\mu_p^{i,j}$ a little bit closer: While the first part $\beta_p^{i,j-1} - \beta_p^{i,j}$ counts the classes that are alive at $\mathbb{M}_i$ and $\mathbb{M}_{j-1}$ but not in $\mathbb{M}_j$ and as such the classes that are alive at $\mathbb{M}_i$ and die entering $\mathbb{M}_j$, the second part $\beta_p^{i-1,j-1} - \beta_p^{i-1,j}$ counts the classes that are alive at $\mathbb{M}_{i-1}$ and $\mathbb{M}_{j-1}$ but no longer at $\mathbb{M}_j$, or equivalently the classes that are alive at $\mathbb{M}_{i-1}$ but die entering $\mathbb{M}_j$. So the difference of those two parts counts the classes that are alive at $\mathbb{M}_i$ and die entering $\mathbb{M}_j$ but were not alive at $\mathbb{M}_{i-1}$, so they are born at $\mathbb{M}_i$. This matches our given interpretation.

Note that while every class gets born at some point (since $\mathbb{M}_0 = \emptyset$), not every class has to die. To be more precise, the classes still alive at $\mathbb{M}_n$ never die, as there are no more steps in our filtration left that could kill them. We call these classes essential:

Definition (Essential homology class)**.** Given persistent homology groups $H_p^{i,j}$ for $0 \leq i, j \leq n$, a homology class $\gamma \in H_p^{i,i}$ is called *essential* iff $\gamma \notin H_p^{i-1,n}$.

Proposition 3.6 (Fundamental lemma of persistent homology)**.** *Let $H_p^{i,j}$ be persistent homology groups of a manifold $\mathbb{M}$ and a Morse function f. For every pair of indices $0 \leq k \leq l \leq n$ and every dimension p, the p-th persistent Betti number is $\beta_p^{k,l} = \beta_p^{k,n} + \sum_{i \leq k} \sum_{j>l} \mu_p^{i,j}$.*

Proof. We simply replace the definition of $\mu_p^{i,j}$ in our sum:

$$\begin{aligned}
\sum_{i\leq k}\sum_{j>l}\mu_p^{i,j} &= \sum_{j>l}\sum_{i\leq k}((\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})) \\
&= \sum_{j>l}\left(\sum_{i\leq k}(\beta_p^{i,j-1} - \beta_p^{i-1,j-1}) - \sum_{i\leq k}(\beta_p^{i,j} - \beta_p^{i-1,j})\right) \\
&= \sum_{j>l}\left((\beta_p^{k,j-1} - \beta_p^{0,j-1}) - (\beta_p^{k,j} - \beta_p^{0,j})\right) \\
&= \sum_{j>l}\left((\beta_p^{k,j-1} - \beta_p^{k,j}) - (\beta_p^{0,j-1} - \beta_p^{0,j})\right) \\
&= (\beta_p^{k,l} - \beta_p^{k,n}) - (\beta_p^{0,l} - \beta_p^{0,n}) \\
&\overset{(*)}{=} \beta_p^{k,l} - \beta_p^{k,n}
\end{aligned}$$

In $(*)$ we recall the definition of $\beta_p^{i,j}$ and see that $\beta_p^{0,j}$ must be 0 for every p and every j, because $K_0 = \emptyset$ cannot contain any homology classes.

By adding $\beta_p^{k,n}$ to this double sum, we get our desired result. □

So what does all this mean contentwise? We were able to prove that all persistent Betti numbers not lasting till $\mathbb{M}_n$ can be reconstructed from the $\mu_p^{i,j}$. That means that these integers contain all significant topological information about our construction using the Morse function, but not about the topology of our final space. This makes sense on the informal level as well: When we know how many homology classes get born at $\mathbb{M}_i$ and die entering $\mathbb{M}_j$, we know how many homology classes are alive at each $\mathbb{M}_k$ that lies in between. We then do not, however, know how many classes never die and thus are counted by $\beta_p^{i,n}$.

Example 3.7. We will illustrate this with an example again: Consider the 2-manifold in figure 3 on the next page. Our Morse function will be the height function relative to the lower edge of the figure. This way, we have exactly 6 critical points $a_1, a_2, \ldots, a_6$, all of them non-degenerate, and choose arbitrary interleaved values $b_0, b_1, \ldots, b_6$.

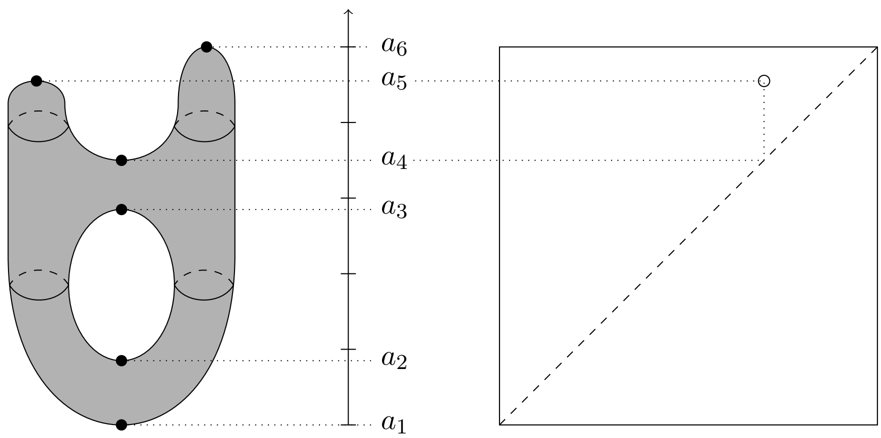

Figure 3: A torus with two "bumps" at the top, unequally high. Also pictured are the critical values of the height function relative to the lower edge of the picture and the accompanying persistence diagram of dimension 1, which contains the only persistent homology class that is born at a_5 and dies at a_6.

We will have a look at how the homology changes when we pass from the sublevel-set $\mathbb{M}_{i-1}$ to $\mathbb{M}_i$ for each $i = 1, 2, \ldots, 6$:

$\mathbb{M}_1$ is homeomorphic to a disc, so $H_p(\mathbb{M}_1) = 0$ except for $p = 0$ where $H_1(\mathbb{M}_1) = \mathbb{Z}_2$.

$\mathbb{M}_2$ is homeomorphic to a tube, so our 0-dimensional class persists and a 1-dimensional class (generated by the loops pictured on either side of the hole in the middle of the figure) is born: $H_1(\mathbb{M}_2) = \mathbb{Z}_2$.

$\mathbb{M}_3$ is homeomorphic to a torus missing one point, so we have yet another 1-dimensional homology class: $H_1(\mathbb{M}_3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$\mathbb{M}_4$ gives rise to a new 1-cycle, generated by the loop through a_3 and a_4: $H_1(\mathbb{M}_4) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$\mathbb{M}_5$ lets the class that was just born at $\mathbb{M}_4$ merge with the class born at $\mathbb{M}_2$, so we only have $H_1(\mathbb{M}_5) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$\mathbb{M}_6$ finally captures the interior space inside the torus and thus gives rise to our only 2-dimensional homology class $H_2(\mathbb{M}_6) = \mathbb{Z}_2$.

We can see that only one homology class was not essential, the one born at $\mathbb{M}_4$ that died at $\mathbb{M}_5$, so the persistence diagram of dimension 1 has only one single dot (those of other dimensions are empty). The distance of this dot from the diagonal of the diagram is the class' persistence, and we can see that it equals the height of the "bump" that caused the class, so now we have a measure for this "artifact" of the original torus, but not for the torus and its hole in general.

3.3 Extended Persistence

To solve the problem of not measuring the final homology, we want every homology class that gets born to die eventually, so that births and deaths always come in pairs. We therefore introduce the concept of *extended persistence*, first mentioned in [EH08] and further described in [CSEH09].

Definition (Extended persistence sequence)**.** Let $\mathbb{M}, f, a_i, b_i, \mathbb{M}_i$ and $\mathbb{M}^i$ be the same as in the definition of ordinary persistent homology groups on page 11. We then recall the sequence

$$0 = H_p(\mathbb{M}_0) \to H_p(\mathbb{M}_1) \to \cdots \to H_p(\mathbb{M}_n).$$

We want to extend this sequence on the right side so that essential homology classes are equipped with a time of death as well. The whole process is illustrated in figure 4 on the following page.

Applying Poincaré duality 2.6 to $H_p(\mathbb{M}_n)$ yields

$$H_p(\mathbb{M}_n) \cong H^{d-p}(\mathbb{M}_n)$$

and by inclusion we have the naturally induced sequence via Lemma 2.4:

$$H_p(\mathbb{M}_n) \cong H^{d-p}(\mathbb{M}_n) \to H^{d-p}(\mathbb{M}_{n-1}) \to \cdots \to H^{d-p}(\mathbb{M}_0) = 0.$$

By applying Lefschetz duality 2.5 to each cohomology group in this sequence, we get

$$H_p(\mathbb{M}_n) \cong H_p(\mathbb{M}_n, \partial\mathbb{M}_n) \to H_p(\mathbb{M}_{n-1}, \partial\mathbb{M}_{n-1}) \to \cdots \to H_p(\mathbb{M}_0, \partial\mathbb{M}_0).$$

Obviously we can replace the boundaries of the sublevel sets by those of the corresponding superlevel sets, as

$$\partial\mathbb{M}_i = \partial\mathbb{M}^i = f^{-1}(\{b_i\})$$

and thus receive the sequence

$$H_p(\mathbb{M}_n, \partial\mathbb{M}^n) \to H_p(\mathbb{M}_{n-1}, \partial\mathbb{M}^{n-1}) \to \cdots \to H_p(\mathbb{M}_0, \partial\mathbb{M}^0).$$

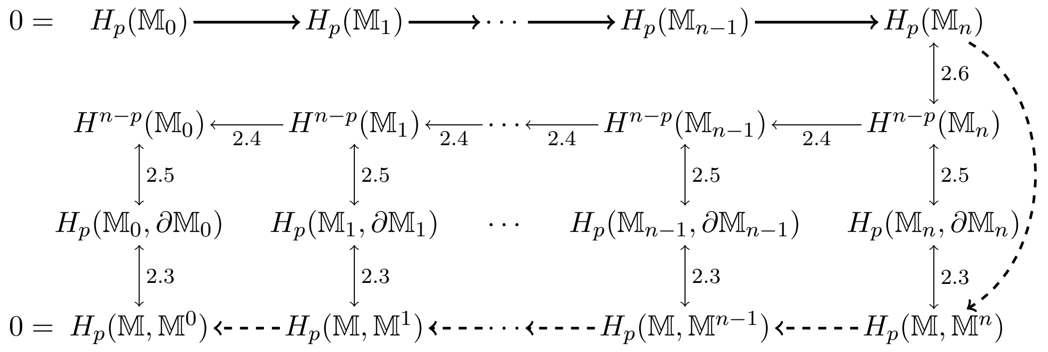

Figure 4: Diagram visualizing the construction of extended persistence on manifolds, pointing out the theorems and lemmas used for each homomorphism. Dashed arrows indicated homomorphisms induced by a commutative sub-diagram, thick arrows highlight the maps that are used later on for the extended persistence sequence.

Finally, we apply the excision theorem (cf. Lemma 2.3) to each homology group to put back in the open set $\operatorname{int}\mathbb{M}^i$ into each pair $(\mathbb{M}_i, \partial\mathbb{M}^i)$, and end up with

$$H_p(\mathbb{M}, \mathbb{M}^n) \to H_p(\mathbb{M}, \mathbb{M}^{n-1}) \to \cdots \to H_p(\mathbb{M}, \mathbb{M}^0).$$

Because of $H_p(\mathbb{M}, \mathbb{M}^n) = H_p(\mathbb{M}, \emptyset) = H_p(\mathbb{M})$, we can append this sequence to our original persistence sequence:

$$\begin{array}{ccccccc} H_p(\mathbb{M}_0) & \longrightarrow & H_p(\mathbb{M}_1) & \longrightarrow & \cdots & \longrightarrow & H_p(\mathbb{M}_n) \\ & & & & & & \| \\ H_p(\mathbb{M}, \mathbb{M}^0) & \longleftarrow & H_p(\mathbb{M}, \mathbb{M}^2) & \longleftarrow & \cdots & \longleftarrow & H_p(\mathbb{M}, \mathbb{M}^n) \end{array}$$

which is the *extended persistence sequence* we searched for.

This extends our sequence known from ordinary persistence, so we can extend our notions for all related terms as well:

Definition (Extended persistent homology groups, extended persistence)**.** Given such an extended persistence sequence we just defined, we get homomorphisms $f_p^{i,j} : X_i \to X_j$ for every pair $0 \le i \le j \le 2n$ where

$$X_i := \begin{cases} H_p(\mathbb{M}_i) & \text{if } i \le n \\ H_p(\mathbb{M}, \mathbb{M}^{2n-i}) & \text{if } i \ge n+1 \end{cases}.$$

Note that in this definition there is only $H_p(\mathbb{M}_n)$ but not $H_p(\mathbb{M}, \mathbb{M}^n)$, because they are equal and no class can be born or die when going from one to the other.

We again call their images the *p-th extended persistent homology groups* $H_p^{i,j} := \operatorname{Im} f_p^{i,j}$ with their ranks the *p-th extended persistent Betti numbers* $\beta_p^{i,j}$ and thus extend our

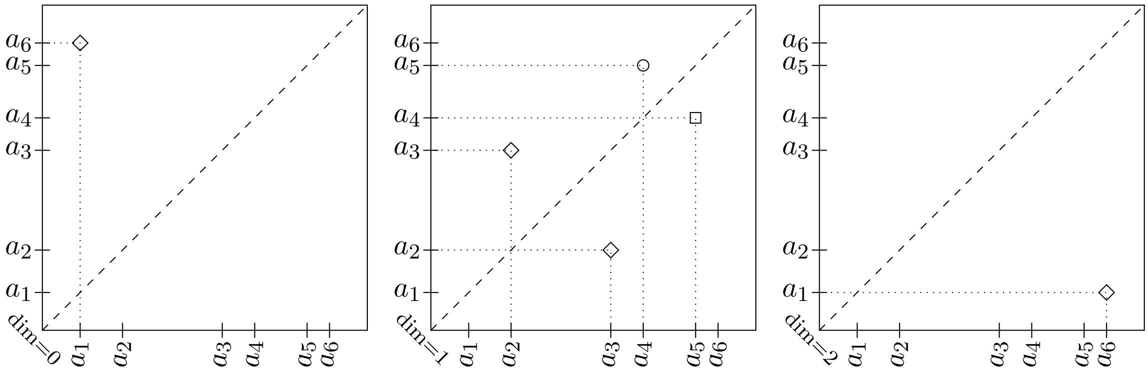

Figure 5: The extended persistence diagrams arising from example 3.7 on page 14 in dimensions 0, 1 and 2. Ordinary classes are depicted as circles, relative classes as squares and extended classes as diamonds.

understanding of birth and death.

When it comes to persistence, however, we do not want to see index persistence greater than n or any persistence smaller than 0. Also, when going from $H_p(\mathbb{M}, \mathbb{M}^i)$ to $H_p(\mathbb{M}, \mathbb{M}^{i-1})$, we actually pass the critical value a_i, unlike before when going from $H_p(\mathbb{M}_i)$ to $H_p(\mathbb{M}_{i+1})$ meant passing the critical value a_{i+1}.

So for a class γ born at $X_i = H_p(\mathbb{M}_i)$ or $X_{2n+1-i} = H_p(\mathbb{M}, \mathbb{M}^{i-1})$ that dies at $X_j = H_p(\mathbb{M}_j)$ or $X_{2n+1-j} = H_p(\mathbb{M}, \mathbb{M}^{j-1})$ with both $0 \leq i, j \leq n$ (but not necessarily $i < j$), we set [1]

$$\text{pers}(\gamma) := |a_j - a_i| \quad \text{and} \quad \text{ipers}(\gamma) := |j - i|.$$

Also, we want to redefine the multiplicities of points in our persistence diagrams appropriately so their size does not increase:

$$\bar{\mu}_p^{i,j} := \mu_p^{i,j} + \mu_p^{i,2n+1-j} + \mu_p^{j,2n+1-i} + \mu_p^{2n+1-i,2n+1-j} \quad \text{for } 0 \leq i, j \leq n.$$

To still be able to distinguish the types of classes, we talk about *ordinary* classes ($i < j \leq n$), *relative* classes ($n < i < j$) and *extended* classes ($i \leq n < j$) and mark them with different symbols in the persistence diagram.

Example 3.8. We continue with our example 3.7 started at page 14 and look at the descending path of relative homology groups, starting with $H_p(\mathbb{M}, \mathbb{M}^6) = H_p(\mathbb{M})$, that we calculated to be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $p = 1$ and $\mathbb{Z}_2$ for $p = 0$ and $p = 2$.

[1] This definition of extended (index) persistence differs from the one given in [CSEH09, sec. 4] because their definition results in unexpected values. In example 3.8, their definition would assign the extended persistence $|a_1 - a_1| = 0$ to the zero-dimensional extended class pictured at (a_1, a_6) in figure 5 which represents the whole torus from figure 3 on page 15, while clearly it should be $|a_1 - a_6|$.

($\mathbb{M}, \mathbb{M}^5$) kills the zero-dimensional class born at $\mathbb{M}_1$ because every point on the surface of our torus is path-connected to a_6 which now lies in $\mathbb{M}^5$.

($\mathbb{M}, \mathbb{M}^4$) glues together the two bumps, generating a hole above a_4 around which a new 1-dimensional homology class is born: $H_1(\mathbb{M}, \mathbb{M}^4) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

($\mathbb{M}, \mathbb{M}^3$) closes the just generated loop and thus kills the according class.

($\mathbb{M}, \mathbb{M}^2$) kills the 1-dimensional class that was born at $\mathbb{M}_2$, because the tube that was born there is now contracted to a single point at the top, so $H_1(\mathbb{M}, \mathbb{M}^2) = \mathbb{Z}_2$.

($\mathbb{M}, \mathbb{M}^1$) buries the last remaining 1-dimensional class, born at $\mathbb{M}_3$ and generated by a loop around the hole in the torus.

($\mathbb{M}, \mathbb{M}^0$) divides every chain out and thus kills the last class, the 2-dimensional one born at $\mathbb{M}_6$.

The resulting diagrams can be seen in figure 5 on the preceding page. We see that the main action takes place in dimension 1, where we now have all three types of classes. Also, we can see an intrinsic symmetry along the main diagonal between the diagrams of dimensions p and $2 - p$. We even have symmetry between fixed types of homology classes: Ordinary classes are always symmetric to relative classes and vice versa, while extended classes are always symmetric to other extended classes.

3.4 Duality and symmetry

The symmetry seen in 3.8 on the previous page is not accidental: It is stated and proven in [CSEH09, sec. 7]:

Theorem 3.9 (Duality theorem, [CSEH09, sec. 7])**.** *Given a d-manifold* $\mathbb{M}$ *and a Morse function* f*, the resulting ordinary, relative and extended persistence diagrams* $\mathrm{Ord}_r(f)$*,* $\mathrm{Rel}_r(f)$ *and* $\mathrm{Ext}_r(f)$ *in dimension* r *are reflections of each other:*

$$\mathrm{Ord}_r(f) = \mathrm{Rel}^T_{d-r}(f),$$
$$\mathrm{Ext}_r(f) = \mathrm{Ext}^T_{d-r}(f),$$

where the superscript T *denotes the reflection along the main diagonal.*

But there is more symmetry to observe: We want to have a quick look at what happens when negating the Morse function.

Example 3.10. We still look at the space from example 3.7, except this time we let the Morse function measure the height relative to the upper edge of the picture in figure 3 on

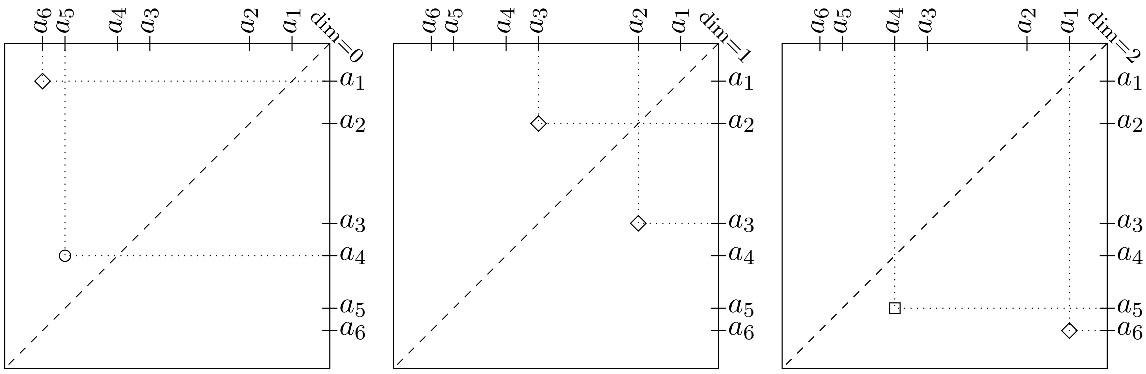

Figure 6: The extended persistence diagrams arising from example 3.7 on page 14 *with negated Morse function* in dimensions 0, 1 and 2. Thus the diagram is still to be read from the lower left corner, even though the origin is in the upper right corner.

page 15. This is the same as if we use our previous Morse function f and negate it to $-f$, because only the ordering of critical values and their distances between each other is what matters for the computation of persistence. The resulting extended persistence diagrams are pictured in figure 6.

Of course we still have the duality across dimensions we just described, but we have another symmetry when comparing the diagrams for f and $-f$: The extended diagrams seem to simply have been reflected and the ordinary and relative diagrams were reflected and shifted in dimension in opposite directions.

The symmetry we get when negating the Morse function is actually a bit more complicated. [CSEH09] describe (and prove) this symmetry in their *Symmetry Theorem* as follows:

> For a real-valued function f on a d-manifold, we have
>
> $$\begin{aligned} \mathrm{Ord}_r(f) &= \mathrm{Ord}^R_{d-r-1}(-f), \\ \mathrm{Ext}_r(f) &= \mathrm{Ext}^0_{d-r}(-f), \\ \mathrm{Rel}_r(f) &= \mathrm{Rel}^R_{d-r-1}(-f), \end{aligned}$$
>
> for all dimensions r.

The superscript R denotes reflection across the minor diagonal, mapping (x, y) to $(-y, -x)$, and the superscript 0 indicates reflection through the origin.

The proof of their version of the symmetry theorem, however, already incorporates the duality noticed earlier. We want to state a version that does not imply the duality theorem:

Theorem 3.11 (Symmetry theorem). *Given a manifold* $\mathbb{M}$*, a Morse function* f *and its negative* $-f$*, the resulting ordinary, relative and extended persistence diagrams* $\mathrm{Ord}_r(f)$*,* $\mathrm{Rel}_r(f)$ *and* $\mathrm{Ext}_r(f)$ *are correlated with the ones for* $-f$*:*

$$\mathrm{Ord}_r(f) = \mathrm{Rel}^0_{r+1}(-f),$$
$$\mathrm{Ext}_r(f) = \mathrm{Ext}^R_r(-f),$$

where the superscript 0 *denotes the reflection through the origin, mapping* (x, y) *to* $(-x, -y)$ *and* R *the reflection along the minor diagonal, mapping* (x, y) *to* $(-y, -x)$*.*

Proof. The proof of this theorem is adapted from a case analysis done in [CSEH09, sec. 4].

First of all we note that the critical points of $-f$ are the same as those of f, only that their values are negated and thus their ordering is reversed. Assume f has the critical points $p_1, p_2, \ldots, p_n \in \mathbb{M}$ with values $a_i = f(p_i)$, and we choose interleaved values $a_0 < b_1 < a_1 < b_2 < a_2 < \ldots < a_n < b_n$ as we did for the definition of the extended persistence sequence. For $-f$, we will call the critical values c_i and have the relation $c_i = -a_{n-i}$. Thus the interleaving values for $-f$ can be chosen like

$$b_n < c_1 < b_{n-1} < c_2 < \ldots < c_n < b_0$$

which gives us the relation

$$\mathbb{M}_i := f^{-1}((-\infty, b_i]) = -f^{-1}([-b_i, \infty)) =: -\mathbb{M}^{n-i}$$

between the sublevel sets of f and the superlevel sets of $-f$. Of course, $\mathbb{M}^i = -\mathbb{M}_{n-i}$ follows analogously. The minus sign in front of $\mathbb{M}$ is merely a notation for indicating that these sub- and superlevel sets are based on $-f$ instead of f.

This means that the extended persistence sequence for $-f$ is

$$\begin{array}{ccccccc} 0 = & H_p(\mathbb{M}^n) & \longrightarrow & H_p(\mathbb{M}^{n-1}) & \longrightarrow \cdots \longrightarrow & H_p(\mathbb{M}^0) \\ & & & & & \| \\ 0 = & H_p(\mathbb{M}, \mathbb{M}_n) & \longleftarrow & H_p(\mathbb{M}, \mathbb{M}_{n-1}) & \longleftarrow \cdots \longleftarrow & H_p(\mathbb{M}, \mathbb{M}_0) \end{array}$$

To relate the diagrams of f and $-f$, we need to state rules that relate this sequence to the one of f. We will do this by analysing the kernels and cokernels of the maps $f^{i,n}_p : H_p(\mathbb{M}_i) \to H_p(\mathbb{M})$.

Fix the i and p and let

$$\begin{aligned} K_p^i &:= \operatorname{Ker} f_p^{i,n} \\ C_p^i &:= H_p(\mathbb{M}) / \operatorname{Im} f_p^{i,n} \\ k &: K_{p-1}^{i-1} \to K_{p-1}^i \\ c &: C_p^{i-1} \to C_p^i \end{aligned}$$

be the maps on the kernels and cokernels induced by the inclusions $\mathbb{M}_{i-1} \hookrightarrow \mathbb{M}_i \hookrightarrow \mathbb{M}$. K_p^i consist of all inessential homology classes in $H_p(\mathbb{M}_i)$, those that die before reaching $\mathbb{M}$, while C_p^i consists of all essential homology classes that have no pre-image in $H_p(\mathbb{M}_i)$ and thus are born later than $\mathbb{M}_i$.

With the help of the long exact sequence for the pair (cf. [Hat12, thm. 2.13]),

$$\ldots \to H_p(\mathbb{M}_i) \to H_p(\mathbb{M}) \to H_p(\mathbb{M}, \mathbb{M}_i) \to H_p(\mathbb{M}_{i-1}) \to \ldots$$

we get the short exact sequence

$$0 \to C_p^i \to H_p(\mathbb{M}, \mathbb{M}_i) \to K_{p-1}^i \to 0$$

which lets the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_p^{i-1} & \xrightarrow{\psi^{i-1}} & H_p(\mathbb{M}, \mathbb{M}_{i-1}) & \xrightarrow{\lambda^{i-1}} & K_{p-1}^{i-1} & \longrightarrow & 0 \\ & & \downarrow c & & \downarrow \varphi & & \downarrow k & & \\ 0 & \longrightarrow & C_p^i & \xrightarrow{\psi^i} & H_p(\mathbb{M}, \mathbb{M}_i) & \xrightarrow{\lambda^i} & K_{p-1}^i & \longrightarrow & 0 \end{array}$$

We now distinguish between four possible cases what the kernel and cokernel of k and c may be like:

Case D.1 $\operatorname{Ker} k \neq 0$. This means that $H_{p-1}(\mathbb{M}_i)$ contains one inessential class less than $H_{p-1}(\mathbb{M}_{i-1})$ and thus this class must have died in this step, lowering the rank of K_{p-1}^i by one compared to K_{p-1}^i. Because we are using homology with coefficients from $\mathbb{Z}_2$, the short exact sequences split and give rise to an isomorphism $H_p(\mathbb{M}, \mathbb{M}_i) = C_p^i \oplus K_{p-1}^i$ which implies that $H_p(\mathbb{M}, \mathbb{M}_i)$ has a lower rank as well (because $C_p^i \cong C_p^{i-1}$, see note on absence of fourth case below):

$$\operatorname{rank} H_p(\mathbb{M}, \mathbb{M}_i) = \operatorname{rank} H_p(\mathbb{M}, \mathbb{M}_{i-1}) - 1.$$

Case D.2 CoKer $k \neq 0$. That means that $H_p(\mathbb{M}_i)$ contains one inessential class more than $H_p(\mathbb{M}_{i-1})$ and thus this class must have been born in this step. It follows that k is not onto, but because λ^{i-1} and λ^i are onto, the commutativity of the diagram forces ϕ to have a non-trivial cokernel, thus raising the rank of the relative homology groups:

$$\operatorname{rank} H_p(\mathbb{M}, \mathbb{M}_i) = \operatorname{rank} H_p(\mathbb{M}, \mathbb{M}_{i-1}) + 1.$$

Case D.3 Ker $c \neq 0$. In this case, $H_p(\mathbb{M}_i)$ contains an essential homology class that was not contained in $H_p(\mathbb{M}_{i-1})$, so this class was born in this step. Because ψ^{i-1} and ψ^i are injective, the commutativity requires φ to be non-injective and thus have non-trivial kernel, lowering the rank of the relative homology groups:

$$\operatorname{rank} H_p(\mathbb{M}, \mathbb{M}_i) = \operatorname{rank} H_p(\mathbb{M}, \mathbb{M}_{i-1}) + 1.$$

As c is always surjective (essential classes never die), these are all possible cases. Only one of them applies because of corollary 3.3 on page 10.

These observations justify the following rules, taken from [EH10, sec. VII.3]:

Rule 1: A dimension p homology class of $\mathbb{M}_i$ dies at the same time that a dimension $p+1$ relative homology class of $(\mathbb{M}, \mathbb{M}_i)$ dies.

Rule 2: An inessential dimension p homology class of $\mathbb{M}_i$ gets born at the same time that a dimension $p+1$ relative homology class of $(\mathbb{M}, \mathbb{M}_i)$ gets born.

Rule 3: An essential dimension p homology class of $\mathbb{M}_i$ gets born at the same time that a dimension p relative homology class of $(\mathbb{M}, \mathbb{M}_i)$ dies.

Rules 1 and 2 directly lead to the first set of equations from the theorem, $\mathrm{Ord}_p(-f) = \mathrm{Rel}^0_{p+1}(f)$, because the ordinary sequence for $-f$ is thereby linked to the relative one of f. The reflection through the origin is simply caused by the reversing of the ordering of critical values.

Rule 3 generates the second set of equations, $\mathrm{Ext}_p(-f) = \mathrm{Ext}^R_p(-f)$: The second coordinate of a point in the first diagram represents the death of a relative homology class and is mapped to the birth of an essential homology class via rule 3. The first coordinate in the first diagram on the other hand represents the birth of an essential class and is thus mapped to the death of a relative class in the second diagram via rule 3. The change of signs is again caused by the reversed ordering. □

We easily see that the symmetry theorem from [CSEH09] quoted above is merely a direct conclusion from applying the duality theorem 3.9 to this version of the symmetry theorem.

4 Persistence on simplicial complexes

In this last section we will try to transfer the definitions of persistence and extended persistence from manifolds to simplicial complexes in general and inspect which properties survive and which properties are lost.

While for manifolds we needed Morse theory to find sequences of ascending spaces to be able to say something about their homologies, we will allow other such "filtrations" on simplicial complexes.

Definition (Filtrations)**.** Let K be a simplicial complex. We call every sequence of increasing subcomplexes $\emptyset = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K$ a *filtration* of K.

Definition (Persistent homology groups, Birth and Death)**.** Let K be a simplicial complex and let $\emptyset = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K$ be a filtration. Then we get an inclusion map $f^{i,j} : K_i \to K_j$ for every $i \leq j$ and thus an induced homomorphism $f_p^{i,j} : H_p(K_i) \to H_p(K_j)$ for every dimension p as per Lemma 2.1.

We will call the images of these homomorphisms $H_p^{i,j} := \mathrm{Im} f_p^{i,j}$ the *p-th persistent homology groups* and their ranks accordingly the *p-th persistent Betti numbers* $\beta_p^{i,j}$.

This is the same setting as for manifolds, so we define *birth*, *death*, $\mu_p^{i,j}$ and *index persistence* in just the same way as we did for manifolds on page 11.

We extended persistence on manifolds by the use of the two dualities of Poincaré and Lefschetz, but these dualities do not apply to simplicial complexes in general. But what resulted from the use of the dualities were the simple homomorphisms $H_p(\mathbb{M}, \mathbb{M}^i) \to H_p(\mathbb{M}, \mathbb{M}^{i-1})$ that are given by inclusion $\mathbb{M}^i \hookrightarrow \mathbb{M}^{i-1}$.

The only thing we need to find a simplicial equivalent to extended persistence are the upper-sets of the Morse function, $\mathbb{M}^i$. Therefore, we introduce special filtrations:

Definition (Vertex ordering filtration)**.** Given a total ordering $v_1, \ldots, v_n$ of the vertices in a simplicial complex K, we can build subcomplexes K_i only having simplices spanned by at most the first $i \in \{0, \ldots, n\}$ vertices:

$$K_i := \{\sigma \in K \mid \sigma \text{ is spanned by a subset of } v_1, v_2, \ldots, v_i\}.$$

These form a filtration which we will call the *(ascending) vertex ordering filtration.*

Every such filtration is paired with the filtration induced by the reverse vertex ordering, called the *descending vertex ordering filtration* and denoted

$$L_i := \{\sigma \in K \mid \sigma \text{ is spanned by a subset of } v_{i+1}, v_{i+2}, \ldots, v_n\}.$$

To make this very clear, we explicitly set $K_0 = L_n = \emptyset$.

Definition (Extended persistent homology groups)**.** Given a simplicial complex K of dimension d and vertex ordering filtrations $K_1 \subsetneq K_2 \subsetneq \ldots \subsetneq K_n$ and $L_n \subsetneq L_{n-1} \subsetneq \ldots \subsetneq L_0$, we construct for each dimension p the sequence

$$\begin{array}{ccccccc} 0 = & H_p(K_0) & \longrightarrow & H_p(K_1) & \longrightarrow \cdots \longrightarrow & H_p(K_n) \\ & & & & & \parallel \\ 0 = & H_p(K, L_0) & \longleftarrow & H_p(K, L_1) & \longleftarrow \cdots \longleftarrow & H_p(K, L_n) \end{array}$$

Analogously to the definitions for manifolds on page 17, we extend the concept of birth, death, $\mu_p^{i,j}$ and persistence to the extended sequence: Let

$$X_i := \begin{cases} H_p(K_i) & \text{if } i \leq n \\ H_p(K, L_{2n-i}) & \text{if } i \geq n+1 \end{cases}$$

and set $\operatorname{pers}(\gamma) := |a_j - a_i|$ and $\operatorname{ipers}(\gamma) := |j - i|$ for a class γ born at $X_i = H_p(K_i)$ or $X_{2n+1-i} = H_p(K, L_{j-1})$ that dies at $X_j = H_p(K_j)$ or $X_{2n+1-j} = H_p(K, L_{j-1})$ with $0 \leq i, j \leq n$.

This definition of extended persistence comes without the use of any duality theorems and even without a function, but instead relies on an ordering of the vertices. This ordering may of course be induced by a smooth function on the underlying space of the simplicial complex which may even be a height function. As a simplicial complex only changes its homology at vertices, and we are only considering finite simplicial complexes, the function does not need to be Morse. An extension to infinite simplicial complexes is of course possible if we require a finite filtration instead.

Example 4.1. We will have a look at the simplicial case with a simple example, pictured in figure 7 on the next page, that is neither a manifold nor a manifold with boundary. The ordering of the vertices is given by their indices.

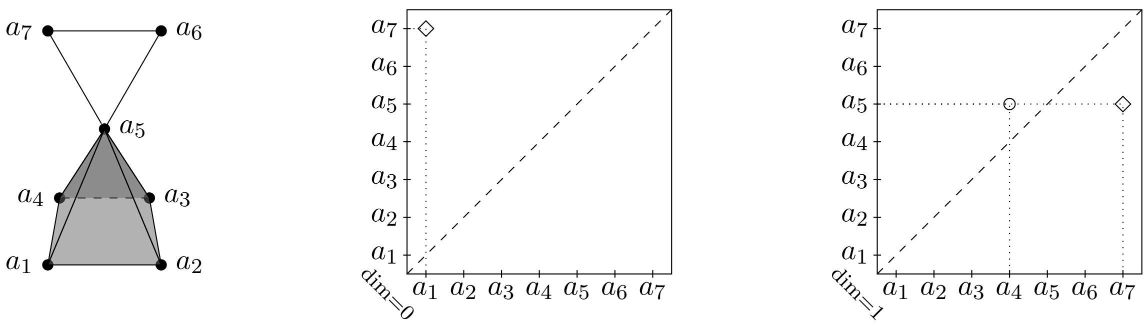

Figure 7: On the left: A simplicial complex consisting of an empty pyramid (without its square base surface but with its lateral surface) with an empty triangle glued at the pyramid's apex. In the middle and on the right: The two according extended persistence diagrams in dimensions 0 and 1.

We can see that $H_p(K_1) = H_p(K_2) = H_p(K_3) = \mathbb{Z}_2^{\delta_{p,0}}$ because the subcomplex is always contractable to a point. Only adding the fourth vertex a_4 gives rise to a 1-dimensional homology class, that is directly killed when adding the fifth vertex a_5 which makes K_5 homeomorphic to a closed disc. a_6 does not change anything about that, only a_7 adds another 1-dimensional class.

At this point we already have the ordinary persistence for all non-essential classes, namely the one born at a_4 and killed at a_5. While its index persistence is obviously 1, we could now define a smooth function on the complex/vertices that is compatible with our given ordering – in this case for example some kind of height function would do.

We continue to observe the extended persistence sequence: $H_0(K, L_1) \cong H_0(K, L_2)$ lack the class born at a_1 that corresponds to the connected component. $H_1(K, L_3)$ misses the 1-cycle born when adding a_7. The rest of the sequence is identical to $H_p(K, L_3)$ as we already killed all homology classes of all dimensions and no relative classes are born.

This results in the persistence diagrams in figure 7. Notice that they are no longer symmetric across dimensions. This was to be expected, as we no longer use the duality theorems for construction. Neither does every index give birth to or kill a homology class of some dimension as it did for manifolds. The reason for that is that a simplicial complex does not change homology at every vertex, while our construction for manifolds did as a result of using Morse functions.

While the simplicial complex in the just discussed example was far from being similar to a manifold, we will have a look at another, more manifold-like example:

Example 4.2. Let K be the triangulation of a pinched torus pictured in figure 8 on the next page. We order the vertices starting with the single contracted point and then going

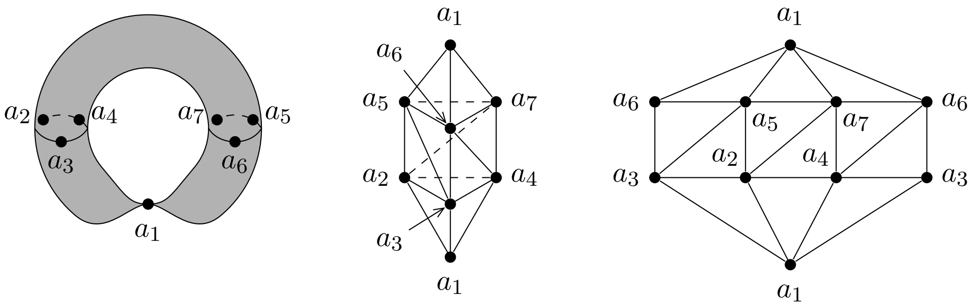

Figure 8: Pinched torus as three dimensional shape and as a triangulation, once glued together along the edge a_1, a_3, a_6, a_1 and once unfolded.

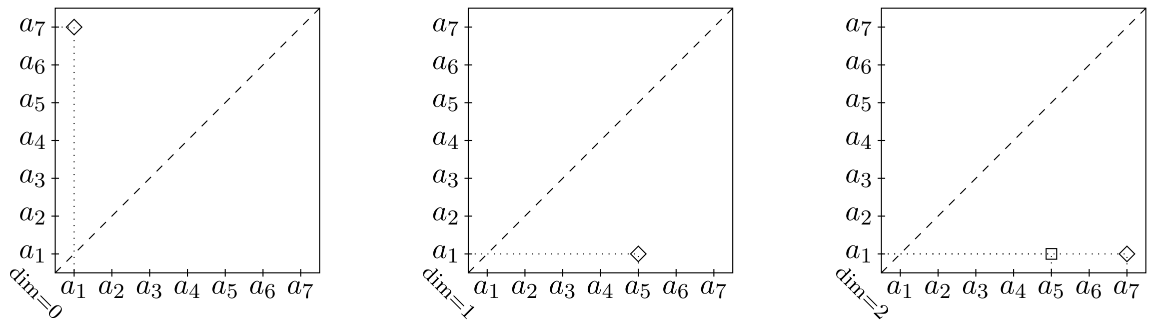

Figure 9: Persistence diagrams in dimensions 0, 1 and 2 arising from the vertex ordering filtration of the simplicial complex in Figure figure 8.

round the torus, like the labels in the figure indicate.

We will not explicitly compute the homology this time, as we have done that often enough, but instead only give the resulting persistence diagrams for discussion in figure 9.

Again we notice that they are not as symmetric as they used to be for manifolds, but they are closer to that symmetry than our previous example.

We have seen that for manifolds, negating the Morse function leads to diagrams that are linked to the original ones. We want to test this for simplicial complexes as well, so we invert the ordering of vertices as shown in figure 10 on the next page and receive the diagrams which are pictured as well.

Surprisingly, with this reversed vertex ordering, we have symmetry across dimensions again — at least if we treat all types of classes equally, unlike we did earlier. Recall that for manifolds, ordinary classes were always reflected to relative classes, and extended classes were reflected to other extended classes. Now we also have symmetry between ordinary and extended classes in dimension 1.

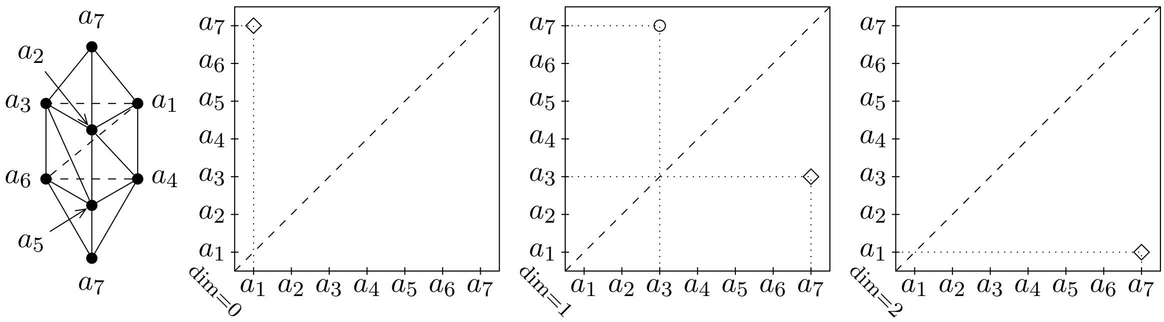

Figure 10: Triangulation of pinched torus from figure 8 on the previous page with reversed vertex ordering along with the resulting extended persistence diagrams.

More than that, our symmetry theorem 3.11 holds for this example. This is reasonable, as the proof does not use the duality theorems, but relies only on the kernels and cokernels of the inclusion-induced homomorphisms between the homology groups. In fact, the original case analysis in [CSEH09] even uses simplicial complexes instead of manifolds.

5 Conclusions

The concept of persistent homology can be extended to include essential homology classes, enabling its application as a measurement tool for topological features of manifolds, be it essential or inessential.

This extension originally was done by the use of Poincaré and Lefschetz duality, which led to an intrinsic symmetry of the persistence diagrams across dimensions for a fixed Morse function. Another symmetry between diagrams of two Morse functions that are each others negative is independent of these duality theorems, but combining them leads to a more convenient symmetry when negating Morse functions.

Without using the duality theorems, a reasonable definition of the extension can be given on simplicial complexes as well. However, apart from the less convenient symmetry between diagrams of mutual negative Morse functions, the properties observable on manifolds cannot be reproduced.

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